Generalized dc and ac Josephson effects in antiferromagnets and in antiferromagnetic *d*-wave superconductors

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The Josephson effect is generally described as Cooper-pair tunneling but it can also be understood in a more general context. The dc Josephson effect is the pseudo-Goldstone boson of two coupled systems with a broken continuous Abelian $U(1)$ symmetry. Hence, an analog should exist for systems with broken continuous non-Abelian symmetries. To exhibit the generality of the phenomenon and make predictions from a realistic model, we study tunneling between antiferromagnets and also between antiferromagnetic *d*-wave superconductors. Performing a calculation analogous to that of Ambegaokar and Baratoff for the Josephson junction, we find an equilibrium current of the staggered magnetization through the junction that, in antiferromagnets, is proportional to $\hat{\mathbf{s}}_L \times \hat{\mathbf{s}}_R$, where $\hat{\mathbf{s}}_L$ and $\hat{\mathbf{s}}_R$ are the Néel vectors on either sides of the junction. Microscopically, this effect exists because of the coherent tunneling of spin-one particle-hole pairs. In the presence of a magnetic field which is different on either sides of the junction, we find an analog of the ac Josephson effect where the angle between Néel vectors depends on time. In the case of antiferromagnetic *d*-wave superconductors we predict that there is a contribution to the critical current that depends on the antiferromagnetic order and a contribution to the spin-critical current that depends on superconducting order. The latter contributions come from tunneling of the triplet Cooper pair that is necessarily present in the ground state of an antiferromagnetic *d*-wave superconductor. All these effects appear to leading order in the square of the tunneling matrix elements.

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I. INTRODUCTION

One of the most striking manifestations of superconductivity is the Josephson effect, which consists in the coherent tunneling of Cooper pairs across a junction when both sides are superconductors. Interestingly, this occurs even though the Hamiltonian contains only single-electron tunneling. The resulting current is proportional to $\sin(\varphi)$, where φ is the difference between the phases of the two superconducting (SC) order parameters. In the presence of a constant potential difference, gauge invariance requires the phase difference to increase linearly with time, leading to an alternating current whose frequency depends only on universal physical constants and not on any material parameter.

Superconductivity is just one example of coherence and spontaneous symmetry breaking. The order parameter is a complex number so the broken symmetry is $U(1)$. The superconducting state selects a phase and the Josephson effect arises from the tendency to make the phase uniform across the tunnel junction.

Since there are many other types of order and corresponding broken symmetries, the question of the analog of the Josephson effect in such cases arises naturally.[1] Indeed, one should expect differences in order parameters across a junction to lead to a coherent tunneling of the condensed objects that exist in the broken symmetry state. This question is especially relevant in the context where junctions between magnetic materials are of utmost importance for spintronics. As a matter of fact, theoretical predictions have been made concerning the possible existence of equilibrium spin currents[2,3] of the Josephson type in ferromagnetic (FM) tunnel junctions, in analogy with superconducting junctions.[4–6] FM long-range order is a realization of spontaneous SO(3) symmetry breaking and an equilibrium spin current would result from the exchange coupling between the magnetic moments in the two leads, which favors alignment of order parameters. From a Ginzburg-Landau point of view, there is a term proportional to $\mathbf{M}_L \cdot \mathbf{M}_R$, where $\mathbf{M_L}$ and $\mathbf{M_R}$ are, respectively, the magnetic moments on the left and on the right of the junction. The Heisenberg equations of motion thus lead to $\frac{d\mathbf{M_L}}{dt} \sim \mathbf{M_L} \times \mathbf{M_R}$, corresponding to a spin current.

In this paper, we study Josephson-type phenomena between antiferromagnets (AF), normal, and superconducting. Such phenomena are interesting for several reasons. In the context of spintronics, it has been shown experimentally that the absence of a net angular momentum in AF results in orders of magnitude faster spin dynamics than in FM, which could expand the now-limited set of applications for AF materials.[7] More generally, in AF the Néel order parameter breaks both lattice translation symmetry and SO(3) spin-rotation symmetry so the situation is less straightforward than in FM. In addition, AF are often close to superconducting phases, as is found in heavy fermions, high-temperature superconductors and layered organic superconductors. Understanding Josephson-type phenomena in AF is the first step toward more general studies with coexisting antiferromagnetic and superconducting order parameters, which we also pursue in this paper. Generalized Josephson effects may help identify homogeneous coexistence of antiferromagnetism and superconductivity in real materials. Studies along these lines have recently appeared for ferromagnetism coexisting with *p*-wave superconductivity.[8]

We begin in Sec. II from a microscopic mean-field model for itinerant antiferromagnets and for single-electron tunneling and establish analogies with the BCS ground state. This microscopic calculation leads in Sec. III to an explicit expression for the analog of the critical current and its temperature dependence. In addition, we show that (a) Cooper-pair tunneling is replaced by tunneling of a spin-one neutral particle-hole pair and (b) time dependence introduced by external magnetic fields resemble the ac Josephson effect but there are many differences because of the non-Abelian nature of the problem and because spins do not couple to the gauge field but directly to the magnetic field. We also discuss in Sec. IV how to observe this effect by coupling to a ferromagnet through a tunnel junction. We then move in Sec. V to the case of d-wave superconducting antiferromagnets and show that there is a contribution to the Josephson charge current that is modulated by the antiferromagnetic order parameter and, conversely, a contribution to the spin-Josephson current that is modulated by the superconducting order parameter. Our results are summarized in Sec. VI. Additional details of the calculation may be found in Ref. 9. For higher-order effects in the tunneling matrix elements[10,11] that we do not discuss here, see also Ref. 9.

II. MODEL

A. Antiferromagnetic state and analogy with BCS ground state

The Hamiltonian for a tunneling junction consisting of two leads of an AF material and an insulating barrier between them reads

$$H = H_L(c^\dagger_{\mathbf{k}\sigma}, c_{\mathbf{k}\sigma}) + H_R(d^\dagger_{\mathbf{q}\sigma}, d_{\mathbf{q}\sigma}) + H_T, \tag{1}$$

where $H_{L(R)}$ is the Hamiltonian of the left (right) AF, H_T is the tunneling part connecting the two leads, $c^\dagger_{\mathbf{k}\sigma}(c_{\mathbf{k}\sigma})$ and $d^\dagger_{\mathbf{q}\sigma}(d_{\mathbf{q}\sigma})$ are the fermion creation (annihilation) operators of the left and right leads, respectively. In the following discussion the quantum numbers $\mathbf{k}$ and $\mathbf{q}$ will also denote implicitly the left and right lead.

We model the AF on each side of the junction by a one-band Hubbard Hamiltonian treated in the Hartree-Fock approximation for a static spin-density wave (SDW) with wave-vector $\mathbf{Q}$. Without loss of generality, we assume the SDW mean field to be polarized along the spin quantization axis. Following Ref. 12, we write

$$\hat{H}_L = \sum_{\mathbf{k}\alpha} \epsilon_k c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} - \frac{US}{2} \sum_{\mathbf{k}\alpha\beta} c^\dagger_{\mathbf{k+Q}\alpha} \sigma^3_{\alpha\beta} c_{\mathbf{k}\beta}, \tag{2}$$

where ϵ_k is the band dispersion, U the interaction strength, σ^3 the third Pauli matrix, while the order parameter S is defined by $(1/N)\langle \Sigma_{\mathbf{k}\alpha\beta} c^\dagger_{\mathbf{k+Q}\alpha}\sigma^3_{\alpha\beta}c_{\mathbf{k}\beta}\rangle$ with N the number of sites. This one-body Hamiltonian can be diagonalized by the Bogoliubov transformation

$$\gamma^c_{\mathbf{k}\alpha} = u_{\mathbf{k}} c_{\mathbf{k}\alpha} + v_{\mathbf{k}} \sum_\beta (\sigma^3)_{\alpha\beta} c_{\mathbf{k+Q}\beta},$$

$$\gamma^v_{\mathbf{k}\alpha} = v_{\mathbf{k}} c_{\mathbf{k}\alpha} - u_{\mathbf{k}} \sum_\beta (\sigma^3)_{\alpha\beta} c_{\mathbf{k+Q}\beta}. \tag{3}$$

To avoid double counting, $\mathbf{k}$ is restricted to the magnetic zone. The superscripts c and v refer to the conduction and the valence bands split by the exchange Bragg scattering from the SDW. For simplicity, we assume perfect nesting $\epsilon_{\mathbf{k}} = -\epsilon_{\mathbf{k+Q}}$. In this case, the coefficients of the transformation are given by $u^2_{\mathbf{k}} = [\frac{1}{2}(1+\epsilon_{\mathbf{k}}/E_{\mathbf{k}})]$, $v^2_{\mathbf{k}} = [\frac{1}{2}(1-\epsilon_{\mathbf{k}}/E_{\mathbf{k}})]$, $E^2_{\mathbf{k}} = (\epsilon^2_{\mathbf{k}} + \Delta^2)$, where $\Delta = US/2$ is the SDW gap parameter.[13] The diagonalized Hamiltonian is given by $H = \Sigma^*_{\mathbf{k}\alpha} E_{\mathbf{k}}(\gamma^{\dagger c}_{\mathbf{k}\alpha}\gamma^c_{\mathbf{k}\alpha} - \gamma^{\dagger v}_{\mathbf{k}\alpha}\gamma^v_{\mathbf{k}\alpha})$, where $\Sigma^*_{\mathbf{k}}$ means that the sum extends over the magnetic zone. The single-particle energy spectrum is given by $\pm E_{\mathbf{k}}$ and the SDW ground state for a half-filled band is defined by $\gamma^{\dagger v}_{\mathbf{k}\alpha}|\Omega\rangle = \gamma^c_{\mathbf{k}\alpha}|\Omega\rangle = 0$, which may be found by filling the vacuum with valence-band quasiparticles,

$$|\Omega\rangle = \prod^*_{\mathbf{k}\alpha} \left(v_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} - u_{\mathbf{k}} \sum_\beta c^\dagger_{\mathbf{k+Q}\beta}\sigma^3_{\beta\alpha} \right) |0\rangle \tag{4}$$

$$= \prod^*_{\mathbf{k}\alpha} \left(v_{\mathbf{k}} - u_{\mathbf{k}} \sum_\beta c^\dagger_{\mathbf{k+Q}\beta}\sigma^3_{\beta\alpha} c_{\mathbf{k}\alpha} \right) c^\dagger_{\mathbf{k}\alpha}|0\rangle. \tag{5}$$

The last form makes the analogies with the BCS ground state clear. For example, there exists Andreev-type reflections at AF-N interfaces.[14,15] The ground state contains coherent particle-hole pairs. To clarify this, we perform a particle-hole transformation for states that are in the first magnetic Brillouin zone. Recall that destroying an electron in a state, creates a hole in the corresponding time-reversed state. For a spinor this can be achieved by $c_{\mathbf{k}\uparrow} \to h^\dagger_{-\mathbf{k}\downarrow}$ and $c_{\mathbf{k}\downarrow} \to -h^\dagger_{-\mathbf{k}\uparrow}$. The ground state then takes the form

$$|\Omega\rangle = \prod^*_{\mathbf{k}} (v_{\mathbf{k}} - u_{\mathbf{k}} c^\dagger_{\mathbf{k+Q}\uparrow} h^\dagger_{-\mathbf{k}\downarrow})(v_{\mathbf{k}} - u_{\mathbf{k}} c^\dagger_{\mathbf{k+Q}\downarrow} h^\dagger_{-\mathbf{k}\uparrow})|0\rangle_h, \tag{6}$$

where $c_{\mathbf{k+Q}\alpha}|0\rangle_h = 0$ and $h_{\mathbf{k}\alpha}|0\rangle_h = 0$. The particle-hole pair is in a triplet state with vanishing net spin projection along the quantization axis, has no charge and has a wave vector equal to the antiferromagnetic wave vector. In the case of a FM, that wave vector would vanish.

In closing this section, note that for a Néel vector oriented in some arbitrary direction with respect to the quantization axis, the variational wave function, Eq. (5), may be written

$$|\Omega\rangle = \prod^*_{\mathbf{k},\alpha} \left(v_{\mathbf{k}} - u_{\mathbf{k}} \sum_{\beta\delta\gamma} c^\dagger_{\mathbf{k+Q}\delta} U_{\delta\beta}\sigma^3_{\beta\gamma} U^\dagger_{\alpha\gamma} c_{\mathbf{k}\gamma} \right) |0\rangle_h, \tag{7}$$

where

$$U(\theta,\phi) = \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) & -e^{-i\phi/2}\sin(\theta/2) \\ e^{i\phi/2}\sin(\theta/2) & e^{i\phi/2}\cos(\theta/2) \end{pmatrix}. \tag{8}$$

In a BCS superconductor, coherence of the pairs is reflected by the fact that they all come with the same phase $e^{i\phi}$ in the BCS state $\Pi_{\mathbf{k}}(u_{\mathbf{k}} - e^{i\phi} v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle$. The $U(1)$ rotation $e^{i\phi}$, identical for all pairs in this case, has its analog in the SU(2) rotation $U_{\delta\beta}\sigma^3_{\beta\gamma}U^\dagger_{\alpha\gamma} = \hat{\mathbf{s}} \cdot \sigma_{\delta\gamma}$, with $\hat{\mathbf{s}} \equiv (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ that, in the antiferromagnetic case, is applied to all particle-hole pairs in the same way. Although our results apply to any dimension, we often quote wave vectors for two dimensions to simplify the discussion.

B. Tunneling Hamiltonian

We choose to define the spin quantization axis along the direction of the instantaneous staggered magnetic moment $\mathbf{S}$. Since we are interested in the case where the moments of the two AF are noncolinear (the analog of a phase difference), the spin quantization axis will differ on each side of the junction. It is thus necessary to include a unitary transformation in spin space, defined by $\mathbf{U}(\theta,\phi)$ in Eq. (8) above, to account for the fact that a spin up on one side of the junction is not the same as a spin up on the other side of the junction. The angles (θ,ϕ) correspond to the orientation of $\mathbf{S}_R$ of the right AF expressed in the coordinate system of the left side of the junction; in Cartesian coordinates $\mathbf{S}_L=|\mathbf{S}_L|(0,0,1)$ and $\mathbf{S}_R=|\mathbf{S}_R|(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. The annihilation operators transform as follows:

$$d_{\mathbf{q}\sigma}=U_{\sigma\sigma'}\tilde{d}_{\mathbf{q}\sigma'} \tag{9}$$

where $\tilde{d}_{\mathbf{q}\sigma'}$ destroys a particle with spin quantization axis along the direction of the Néel vector on the right-hand side of the junction. The summation over repeated indices is implied. The tunneling Hamiltonian then reads

$$\hat{H}_T=(1/N)\sum_{\mathbf{kq}\sigma\sigma'}(t_{\mathbf{kq}}c^{\dagger}_{\mathbf{k}\sigma}U_{\sigma\sigma'}\tilde{d}_{\mathbf{q}\sigma'}+\text{H.c.}). \tag{10}$$

The spin flip terms come purely from the choice of different quantization axes on the left and on the right. There is no real spin flip in the tunneling process.

By choosing the quantization axis along the Néel vector, we performed the analog of a gauge choice. The choice of a different basis (quantization axis) on either side is reflected completely in the $\mathbf{U}(\theta,\phi)$ appearing in the tunneling Hamiltonian. The angles in the unitary transformation $\mathbf{U}(\theta,\phi)$ are the non-Abelian analog of the phase in the ordinary Josephson effect. In the latter case, electromagnetic fields coupled to charge affect the phase difference $e^{-ie\xi/\hbar c}$ through a gauge transformation $A_\mu\rightarrow A_\mu+\partial_\mu\xi$ on one side of the junction. Since we can write $\mathbf{U}(\theta,\phi)=e^{-i\sigma_z\phi/2}e^{-i\sigma_y\theta/2}$, for example, we see that in our case the gauge fields θ,ϕ are coupled to spins, as is the magnetic field through the Zeeman effect. Such a field $\mathbf{B}$ yields a unitary time evolution $e^{ig\mu_B\mathbf{B}\cdot\sigma t/2}$, where g is the gyromagnetic ratio, μ_B is the Bohr magneton, and t the time. Here it is the physical field and not the gauge field that affects the generators of SU(2) rotations.

III. SPIN JOSEPHSON EFFECT

To obtain the usual Josephson effect in superconductivity, one computes the time evolution of the number operator, which is conjugate to the phase. The phase is the infinitesimal generator of the broken $U(1)$ rotations. For a broken symmetry that is the staggered magnetization along $\hat{\mathbf{z}}$, let us first consider the infinitesimal generator of SU(2) rotations along x, $S^x(\mathbf{q}=0)$. Since the commutator $[S^x(\mathbf{q}=0),S^y(\mathbf{q}=(\pi,\pi))]$ is proportional to the order parameter, a constant, this means that the analog of the number operator in this case is $S^y[\mathbf{q}=(\pi,\pi)]$. By considering $S^y(\mathbf{q}=0)$ for infinitesimal rotations along y, one could have concluded that $S^x[\mathbf{q}=(\pi,\pi)]$ was the analog of the number operator.[5] Furthermore, S^z will change directions. This means that one will find the analog of the Josephson effect, both dc and ac, by writing down the equations of motion for the vector $\mathbf{S}[\mathbf{q}=(\pi,\pi)]$. This is what we proceed to do in this section.

A. Steady state

The staggered magnetic moment operator in the left lead is $\hat{\mathbf{S}}_L=(\hbar/2)\Sigma_{\mathbf{k}\alpha\beta}c^{\dagger}_{\mathbf{k+Q}\alpha}\sigma_{\alpha\beta}c_{\mathbf{k}\beta}$, where σ is a vector of Pauli matrices. In the broken symmetry state, the time evolution of $\hat{\mathbf{S}}_L$ due to H_L is negligible. Since we also have $[\hat{\mathbf{S}}_L,\hat{H}_R]=0$ we find $d\hat{\mathbf{S}}_L/dt=(1/i\hbar)[\hat{\mathbf{S}}_L,\hat{H}_T]$, and thus $d\hat{\mathbf{S}}_L/dt=-(i/2N)\Sigma_{\mathbf{kq}}\Sigma_{\alpha\beta\delta}(\sigma_{\alpha\beta}\mathbf{U}_{\beta\delta}t_{\mathbf{kq}}c^{\dagger}_{\mathbf{k+Q}\alpha}\tilde{d}_{\mathbf{q}\delta}-\text{H.c.})$ whose average $\dot{\mathbf{S}}_L(t)\equiv\langle d\hat{\mathbf{S}}_L/dt\rangle$ is given by

$$\dot{\mathbf{S}}_L(t)=\frac{1}{N}\sum_{\mathbf{kq}}\sum_{\alpha\beta\delta}\text{Im}[\sigma_{\alpha\beta}\mathbf{U}_{\beta\delta}t_{\mathbf{kq}}\langle c^{\dagger}_{\mathbf{k+Q}\alpha}(t)\tilde{d}_{\mathbf{q}\delta}(t)\rangle], \tag{11}$$

where $\langle\cdots\rangle$ is the thermal statistical average with the full density matrix. Performing first-order perturbation theory using $\hat{H}_T$ as the perturbation, one obtains

$$\langle c^{\dagger}_{\mathbf{k+Q}\alpha}\tilde{d}_{\mathbf{q}\delta}\rangle=-\frac{i}{\hbar}\int_{-\infty}^{t}dt'\langle[c^{\dagger}_{\mathbf{k+Q}\alpha}(t)\tilde{d}_{\mathbf{q}\delta}(t),\hat{H}_T(t')]\rangle_0, \tag{12}$$

where the average $\langle\cdots\rangle_0$ is computed with $H_0=\hat{H}_L+\hat{H}_R$ (the unperturbed part of $\hat{H}$). The operators on the right are in the interaction representation.

Once the commutator is evaluated in Eq. (12), one of the factorizations of the four-point correlation function involves products of correlation functions on the left and on the right leads such as $\langle c^{\dagger}_{\mathbf{k+Q}\alpha}(t)c_{\mathbf{k}\alpha}(t')\rangle_0\langle\tilde{d}_{\mathbf{q+Q}\delta}(t)\tilde{d}^{\dagger}_{\mathbf{q}\delta}(t')\rangle_0$. Such correlation functions would vanish in a normal paramagnetic state. They are nonzero because of the broken symmetry. They represent interference in the tunneling process between momentum $\mathbf{k+Q}$ spin-up particles and momentum $-\mathbf{k}$ spin-down holes, in other words tunneling of charge zero spin one $S^z=0$ coherent particle-hole pairs that have finite momentum and are present in the ground state, Eq. (5). In the case of the ordinary Josephson effect, one would find terms such as $\langle c^{\dagger}_{\mathbf{k}\sigma}(t)c^{\dagger}_{-\mathbf{k}-\sigma}(t')\rangle_0\langle\tilde{d}_{\mathbf{q}\sigma}(t)\tilde{d}_{-\mathbf{q}-\sigma}(t')\rangle_0$ that represent tunneling of coherent Cooper pairs.

In order to compute the averages $\langle\cdots\rangle_0$ in the broken-symmetry states, we invert the Bogoliubov transformation Eq. (3). Assuming $t_{\mathbf{kq}}=t_{\mathbf{kq+Q}}=t_{\mathbf{k+Qq}}=t_{\mathbf{k+Qq+Q}}$, we find

$$\sum_{\mathbf{kq}}t_{\mathbf{kq}}c^{\dagger}_{\mathbf{k+Q}\alpha}\tilde{d}_{\mathbf{q}\delta}=\sum_{\mathbf{kq}}^{*}\sum_{i,j\in\{c,v\}}t_{\mathbf{kq}}(\Gamma^{\alpha\delta}_{\mathbf{kq}})_{ij}\gamma^{j\dagger}_{\mathbf{k}\alpha}\gamma^{j}_{\mathbf{q}\delta}, \tag{13}$$

where we defined

$$(\Gamma^{\alpha\delta}_{\mathbf{kq}})_{cc}\equiv(u_{\mathbf{k}}u_{\mathbf{q}}+\sigma^3_{\alpha\alpha}v_{\mathbf{k}}u_{\mathbf{q}}+\sigma^3_{\delta\delta}u_{\mathbf{k}}v_{\mathbf{q}}+\sigma^3_{\alpha\alpha}\sigma^3_{\delta\delta}v_{\mathbf{k}}v_{\mathbf{q}}),$$

$$(\Gamma^{\alpha\delta}_{\mathbf{kq}})_{cv}\equiv(v_{\mathbf{k}}u_{\mathbf{q}}-\sigma^3_{\alpha\alpha}u_{\mathbf{k}}u_{\mathbf{q}}+\sigma^3_{\delta\delta}v_{\mathbf{k}}v_{\mathbf{q}}-\sigma^3_{\alpha\alpha}\sigma^3_{\delta\delta}u_{\mathbf{k}}v_{\mathbf{q}}),$$

$$(\Gamma_{\mathbf{kq}}^{\alpha\delta})_{vc} \equiv (u_{\mathbf{k}}v_{\mathbf{q}} + \sigma_{\alpha\alpha}^{3}v_{\mathbf{k}}v_{\mathbf{q}} - \sigma_{\delta\delta}^{3}u_{\mathbf{k}}u_{\mathbf{q}} - \sigma_{\alpha\alpha}^{3}\sigma_{\delta\delta}^{3}v_{\mathbf{k}}u_{\mathbf{q}}),$$

$$(\Gamma_{\mathbf{kq}}^{\alpha\delta})_{vv} \equiv (v_{\mathbf{k}}v_{\mathbf{q}} - \sigma_{\alpha\alpha}^{3}u_{\mathbf{k}}v_{\mathbf{q}} - \sigma_{\delta\delta}^{3}v_{\mathbf{k}}u_{\mathbf{q}} + \sigma_{\alpha\alpha}^{3}\sigma_{\delta\delta}^{3}u_{\mathbf{k}}u_{\mathbf{q}}). \tag{14}$$

Similarly, let $\tilde{H}_T$ denote the part of $\hat{H}_T$ that does not commute with $c_{\mathbf{k}+\mathbf{Q}\alpha}^{\dagger}\tilde{d}_{\mathbf{q}\delta}$. It can be written as $\tilde{H}_T=(1/N)\Sigma^{*}_{\mathbf{kq}\sigma\delta'}\Sigma_{ij}\mathbf{U}^{*}_{\sigma\delta'}t^{*}_{\mathbf{kq}}(\Gamma_{\mathbf{kq}}^{\sigma\delta'})_{ij}\gamma_{\mathbf{q}\delta'}^{j\dagger}\gamma_{\mathbf{k}\sigma}^{i}$. Substituting these expressions into Eq. (12) and taking into account the fact that to this order in $t_{\mathbf{kq}}$ the unitary transformation $\mathbf{U}(t')$ in the tunneling matrix element can be evaluated at $t'=t$ since $\hat{H}_L$ and $\hat{H}_R$ do not change the quantization axis, one finds

$$\dot{\mathbf{S}}_L(t) = \frac{1}{N^2}\sum_{\mathbf{kq}}^{*}\sum_{\alpha\beta\delta}\sum_{\sigma\delta'}\mathrm{Im}\Bigg\{-\frac{i}{\hbar}|t_{\mathbf{kq}}|^2\vec{\sigma}_{\alpha\beta}\mathbf{U}_{\beta\delta}\mathbf{U}^{*}_{\sigma\delta'}\int dt'\, e^{-0^{+}(t-t')}\times\sum_{ij}(\Gamma_{\mathbf{kq}}^{\alpha\delta})_{ij}(\Gamma_{\mathbf{kq}}^{\sigma\delta'})_{ij}[\mathcal{G}^{i<}_{\mathbf{k}\sigma\alpha}(t'-t)\mathcal{G}^{j>}_{\mathbf{q}\delta\delta'}(t-t') - \mathcal{G}^{i>}_{\mathbf{k}\sigma\alpha}(t'-t)\mathcal{G}^{j<}_{\mathbf{q}\delta\delta'}(t-t')]\Bigg\}, \tag{15}$$

where $\mathcal{G}^{i<(>)}_{\mathbf{k(q)}}$ are the Keldysh Green's functions in the left (right) lead. Their definitions are $\mathcal{G}^{i<}_{\mathbf{k(q)},\alpha\beta}(t,t')=i\langle\gamma^{j\dagger}_{\mathbf{k(q)}\beta}(t')\gamma^{j}_{\mathbf{k(q)}\alpha}(t)\rangle$ and $\mathcal{G}^{i>}_{\mathbf{k(q)},\alpha\beta}(t,t')=-i\langle\gamma_{\mathbf{k(q)}\alpha}(t)\gamma^{j\dagger}_{\mathbf{k(q)}\beta}(t')\rangle$, respectively. Explicitly,

$$\mathcal{G}^{i>}_{\mathbf{k}\sigma\sigma'}(t'-t) = -i[1-f(E_k^i)]\exp[-iE_k^i(t'-t)/\hbar]\delta_{\sigma\sigma'},$$

$$\mathcal{G}^{i<}_{\mathbf{k}\sigma\sigma'}(t'-t) = if(E_k^i)\exp[-iE_k^i(t'-t)/\hbar]\delta_{\sigma\sigma'},$$

where f is the Fermi function. Equation (15) for the staggered magnetic moment current through a tunnel junction is general. A bias could be included. We assume that there is no bias so there is no incoherent single-particle tunneling across the antiferromagnetic gap. Integrating over $t-t'$ and performing the spin sum in Eq. (15), one finds

$$\dot{\mathbf{S}}_L = I_c\hat{\mathbf{s}}_L\times\hat{\mathbf{s}}_R, \tag{16}$$

where $\hat{\mathbf{s}}_{L(R)}=\mathbf{S}_{L(R)}/|\mathbf{S}_{L(R)}|$. To obtain the correct sign, one must take into account in the Fourier transforms that the last site to the left and the first site to the right do not belong to the same sublattice. In the above equation, I_c is defined by

$$I_c = \frac{8\Delta_L\Delta_R}{N^2}P\sum_{kq}^{*}|t_{\mathbf{kq}}|^2\frac{f(E_{\mathbf{k}})-f(-E_{\mathbf{k}})}{E_{\mathbf{k}}(E_{\mathbf{k}}^2-E_{\mathbf{q}}^2)}$$

with P indicating principal part. A similar expression is found for the equilibrium spin current in the case of ferromagnetic tunnel junctions. Note that the sine function present in the standard Josephson case is replaced here by a cross product, which is a direct consequence of the vectorial nature of the order parameter.

For a symmetrical junction ($\Delta_L=\Delta_R$), the same assumptions and procedure as Ref. 16 lead to the following analytical result

$$I_c = \frac{h}{e^2}R^{-1}\Delta(T)\tanh\left[\frac{1}{2}\beta\Delta(T)\right], \tag{17}$$

where $R=\hbar/(4\pi e^2D^2|t|^2)$ is the (zero-temperature) normal-state resistance of the junction with D the density of state, which is assumed to be a constant. This expression for the temperature dependence of the critical current has the same form as that obtained by Ambegaokar and Baratoff[16] for a BCS superconductor, which is not surprising given the formal analogies.[17]

By symmetry, the time derivative of the staggered magnetic moment on the right lead can be obtained by interchanging the L and R indices in Eq. (16). As a consequence, the staggered magnetic moments of the two AF precess about their (constant) sum $\mathbf{S}_L+\mathbf{S}_R$ at a frequency $\omega_0=I_c|\mathbf{S}_L+\mathbf{S}_R|/|\mathbf{S}_L||\mathbf{S}_R|$.

B. ac spin-Josephson effect

In the ordinary Josephson effect, the electromagnetic gauge potentials enter directly in the argument of the sine function. The present case is different. Each magnetic moment associated with a spin couples to the magnetic field through the Zeeman term ($H_Z=-g\mu_B\mathbf{B}\cdot\mathbf{S}$), where g is the gyromagnetic ratio and μ_B the Bohr magneton (we neglect terms coming from orbital motion[18]).

Considering magnetic fields $\mathbf{B}_L$ and $\mathbf{B}_R$ applied, respectively, to the left- and right-hand sides of the junction, the Heisenberg equations of motion lead to the following equations of motion for the order parameters:

$$\dot{\mathbf{S}}_L = -g\mu_B\mathbf{B}_L\times\mathbf{S}_L + I_c\hat{\mathbf{s}}_L\times\hat{\mathbf{s}}_R,$$

$$\dot{\mathbf{S}}_R = -g\mu_B\mathbf{B}_R\times\mathbf{S}_R + I_c\hat{\mathbf{s}}_R\times\hat{\mathbf{s}}_L. \tag{18}$$

The first term on the right side of the equality is merely the contribution of H_Z to the Heisenberg equation of motion. The second term is the tunneling contribution and has exactly the same form as the one we have already computed in the zero-field case, namely, Eq. (16).

To prove the correctness of the $I_c\hat{\mathbf{s}}_L\times\hat{\mathbf{s}}_R$ terms, we must return to the contribution from the first order in perturbation-theory term Eq. (12). The expectation values are now evaluated with the unperturbed Hamiltonian $\hat{H}_L-g\mu_B\mathbf{B}_L\cdot\mathbf{S}+\hat{H}_R-g\mu_B\mathbf{B}_R\cdot\mathbf{S}$. Contrary to the case where no magnetic field was applied, the quantization axes now precess and one must worry about the time dependence of the rotation matrix $U^{\dagger}_{\delta'\sigma'}(t')$ in terms of the form $\tilde{d}^{\dagger}_{\mathbf{q}\delta'}(t')U^{\dagger}_{\delta'\sigma'}(t')c_{\mathbf{k}\sigma'}(t')$. That rotation matrix is no longer evaluated at time t. The creation-annihilation operators are in the interaction representation and the original Hamiltonian commutes with the Zeeman Hamiltonian. Hence, it is possible to factorize the time evolution due to the magnetic fields to obtain $\tilde{d}^{\dagger}_{\mathbf{q}\delta'}(t')=\tilde{d}^{0\dagger}_{\mathbf{q}\delta}(t')\Lambda^{R\dagger}_{\delta\delta'}(t')$ and $c_{\mathbf{k}\sigma'}(t')=\Lambda^{L}_{\sigma'\sigma}(t')c^{0}_{\mathbf{k}\sigma}(t')$, where

$$\Lambda^{R(L)}_{\sigma'\sigma}(t') = \exp[ig\mu_B \mathbf{B}_{R(L)} \cdot \sigma t'/2]_{\sigma'\sigma} \tag{19}$$

and the superscript 0 on creation-annihilation operators indicates time evolution with $\hat{H}_0 = \hat{H}_L + \hat{H}_R$. Since $\Lambda^L_{\sigma'\sigma}(t') = \Lambda^L_{\sigma'\sigma}(t'-t)\Lambda^L_{\sigma\bar{\sigma}}(t)$ we can also write

$$c_{\mathbf{k}\sigma'}(t') = \Lambda^L_{\sigma'\sigma}(t'-t)\Lambda^L_{\sigma\bar{\sigma}}(t)c^0_{\mathbf{k}\bar{\sigma}}(t') \tag{20}$$

and define creation-annihilation operators whose spin basis is rotated by the magnetic field,

$$c^t_{\mathbf{k}\sigma}(t') \equiv \Lambda^L_{\sigma\bar{\sigma}}(t)c^0_{\mathbf{k}\bar{\sigma}}(t')$$

so that

$$c_{\mathbf{k}\sigma'}(t') = \Lambda^L_{\sigma'\sigma}(t'-t)c^t_{\mathbf{k}\sigma}(t'). \tag{21}$$

The superscript t in $c^t_{\mathbf{k}\sigma}(t')$ reminds us that the spin basis is that defined by the Néel vector at time t. That basis rotates because of the magnetic field. Somewhat redundantly, we also use σ and σ' to denote the spin basis on the left of the junction at time t and t', respectively. Physically, the last result tells us that the time evolution of the basis due to the magnetic field can be taken into account separately.

We now need to work out the corresponding transformation laws for the unitary transformation $U^\dagger_{\delta'\sigma'}(t')$. This operator changes the basis from left (σ') to right (δ') at time t'. The only time dependence in $U^\dagger_{\delta'\sigma'}(t')$ comes from that of the basis on the left and on the right due to the magnetic fields, it is not an additional time dependence. Hence, the transformation law, which we prove in more details below, is

$$U^\dagger_{\delta'\sigma'}(t') = \Lambda^R_{\delta'\delta}(t'-t)U^\dagger_{\delta\sigma}(t)\Lambda^{L\dagger}_{\sigma\sigma'}(t'-t), \tag{22}$$

where we let δ and δ' denote the spin basis on the right at times t and t', respectively. Given the transformation law for the operators on the left, Eq. (21), and the analogous one for the operators on the right, we will thus find

$$\tilde{d}^\dagger_{\mathbf{q}\delta'}(t')U^\dagger_{\delta'\sigma'}(t')c_{\mathbf{k}\sigma'}(t') = \tilde{d}^{t\dagger}_{\mathbf{q}\delta}(t')U^\dagger_{\delta\sigma}(t)c^t_{\mathbf{k}\sigma}(t'), \tag{23}$$

i.e., all spin indices are at time t on the right-hand side of the equality. Therefore, it is possible to factor the unitary transformation $U^\dagger_{\delta\sigma}(t)$ out of the integral over t' in the perturbation formula, Eq. (12). The rest of the calculation then just follows the same path as in the zero-field case, leading to the above equations of motion, Eq. (18).

We now complete the proof on the transformation properties of the unitary matrices in Eq. (22). First, express the left-hand side creation operators alternatively in the right- and in the left-hand side basis. More specifically, left multiplying by $U^\dagger$ the change in basis Eq. (9), the annihilation operators of left-hand side can be expressed in the spin basis of the right-hand side

$$U^\dagger_{\delta'\sigma'}(t')c_{\mathbf{k}\sigma'}(t') = \tilde{c}_{\mathbf{k}\delta'}(t'). \tag{24}$$

Now we change to the right-hand side basis at time t with Eq. (21)

$$U^\dagger_{\delta'\sigma'}(t')c_{\mathbf{k}\sigma'}(t') = \Lambda^R_{\delta'\delta}(t'-t)\tilde{c}^t_{\mathbf{k}\delta}(t') \tag{25}$$

and we return to the creation operators in the spin basis appropriate for the left-hand side at time t,

$$U^\dagger_{\delta'\sigma'}(t')c_{\mathbf{k}\sigma'}(t') = \Lambda^R_{\delta'\delta}(t'-t)U^\dagger_{\delta\sigma}(t)\tilde{c}^t_{\mathbf{k}\sigma}(t'). \tag{26}$$

We arrive at our final result by inserting the identity and changing the spin basis from time t to time t', using Eq. (21) again

$$U^\dagger_{\delta'\sigma'}(t')c_{\mathbf{k}\sigma'}(t') = \Lambda^R_{\delta'\delta}(t'-t)U^\dagger_{\delta\sigma}(t) \times[\Lambda^{\dagger L}_{\sigma\sigma'}(t'-t)\Lambda^L_{\sigma'\alpha}(t'-t)]\tilde{c}^t_{\mathbf{k}\alpha}(t') \tag{27}$$

$$=[\Lambda^R_{\delta'\delta}(t'-t)U^\dagger_{\delta\sigma}(t)\Lambda^{\dagger L}_{\sigma\sigma'}(t'-t)]c_{\mathbf{k}\sigma'}(t'). \tag{28}$$

Comparing the left- and right-hand sides, this completes the proof of the transformation property for the change in basis from left to right, Eq. (23) at different times.

We close this section by solving and interpreting the solution of the equations of motion, Eq. (18). If $\mathbf{B}_L = \mathbf{B}_R = \mathbf{B}$, one finds that in the rotating frame $\hat{\mathbf{u}} = \{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$ defined by $d\hat{\mathbf{u}}/dt = -g\mu_B \mathbf{B} \times \hat{\mathbf{u}}$, $\mathbf{S}_L$ and $\mathbf{S}_R$ still precess about their (constant) sum $\mathbf{\Sigma} \equiv \mathbf{S}_L + \mathbf{S}_R$ at a frequency ω_0. Returning to the static frame, Eq. (18) gives $d\mathbf{\Sigma}/dt = -g\mu_B \mathbf{B} \times \mathbf{\Sigma}$ so that $\mathbf{S}_L$ and $\mathbf{S}_R$ undergo a motion of double precession.

In the ordinary Josephson effect, a constant electric potential difference V leads to a phase difference φ that depends linearly on time ($\dot{\varphi} = 2eV/\hbar$). In the present case, an analog is found by computing the time dependence of the relative orientation ϑ between $\mathbf{S}_L$ and $\mathbf{S}_R$. Using Eq. (18) to compute $d(\cos\vartheta)/dt = d(\hat{\mathbf{s}}_L \cdot \hat{\mathbf{s}}_R)/dt$, we obtain

$$\dot{\vartheta}(t) = -g\mu_B \delta\mathbf{B} \cdot \hat{\mathbf{e}}(t), \tag{29}$$

where $\delta\mathbf{B} \equiv \mathbf{B}_L - \mathbf{B}_R$ and $\hat{\mathbf{e}}(t) \equiv \hat{\mathbf{s}}_R(t) \times \hat{\mathbf{s}}_L(t)$. As in the ordinary Josephson effect, this expression does not contain explicitly the tunneling matrix element. Contrary to the ordinary Josephson effect, this equation is nonlinear. Solving it numerically along with Eq. (18) in the case where the magnetic field vanishes on one side of the junction, one finds that the angle between $\mathbf{S}_L$ and $\mathbf{S}_R$ behaves as a sine-like function of time. The presence of an additional constant magnetic field $\mathbf{B}$ throughout the system adds a beat to this sine-like behavior. The gyromagnetic ratio g is material dependent. No such nonuniversal constant appears in the ordinary Josephson effect. The above discussion of the ac effect can be transposed for FM by replacing the staggered magnetic moment by the uniform one.

IV. EXPERIMENTAL DETECTION

In principle, to detect the effects described above, one could use antiferromagnetic resonance.[19] This phenomenon is based on an anisotropy field resulting from the spin-orbit interaction. However the effect is absent for a spin 1/2. Assuming however that the same equations would hold for $S=1$, it becomes possible to detect the effect by antiferromagnetic resonance.[19] In the limit where $\mathbf{S}_L$ and $\mathbf{S}_R$ are nearly colinear, the effect should therefore lead to a uniform shift of order ω_0 in the antiferromagnetic resonance frequencies of each AF. With $|\frac{2\Delta}{S}|=U=2$ eV and a normal-state conductance on the order of the conductance quantum, $R^{-1}=2e^2/h$, one finds a value of ω_0 of order 10^{14} Hz, in other words in the visible. This frequency would be higher than the single-particle gap and so would lead to much damping. Resistances that are many orders of magnitude larger are thus needed to bring its value down.

To detect the spin-Josephson effect even in the spin-1/2 case, we suggest to add an additional tunnel junction to a ferromagnet. In other words, consider an AF/I/AF/I/FM junction. Indeed, let us begin just with the AF/I/FM junction. The ferromagnetic material can be described by the Stoner Hamiltonian

$$H=(d^{\dagger}_{\mathbf{q}\uparrow}\quad d^{\dagger}_{\mathbf{q}\downarrow})\begin{pmatrix}\zeta_{\mathbf{q}}-h & 0\\ 0 & \zeta_{\mathbf{q}}+h\end{pmatrix}\begin{pmatrix}d_{\mathbf{q}\uparrow}\\ d_{\mathbf{q}\downarrow}\end{pmatrix}, \tag{30}$$

where $\zeta_{\mathbf{q}}=\varepsilon_{\mathbf{q}}-\mu$ and h is a molecular field. Using the same tunneling Hamiltonian as before and following the same procedure, we find that

$$\dot{\mathbf{S}}_L=\mathcal{I}_c\mathbf{M}_R\times\mathbf{S}_L, \tag{31}$$

$$\dot{\mathbf{M}}_R=\mathcal{I}_c\mathbf{S}_L\times\mathbf{M}_R, \tag{32}$$

where $\mathbf{M}_R$ is the average magnetization of the ferromagnet and

$$\mathcal{I}_c=\frac{1}{N^2}\mathcal{P}\sum_{\mathbf{k}}^{*}\sum_{\mathbf{q}}|t_{\mathbf{k}q}|^2\frac{\Delta_L}{E_{\mathbf{k}}}\Bigg\{\frac{f(\zeta_{\mathbf{q}}-h)-f(E_{\mathbf{k}})}{\zeta_{\mathbf{q}}-h-E_{\mathbf{k}}}-\frac{f(\zeta_{\mathbf{q}}-h)-f(-E_{\mathbf{k}})}{\zeta_{\mathbf{q}}-h+E_{\mathbf{k}}}-(h\to-h)\Bigg\}. \tag{33}$$

Since the uniform magnetization of the ferromagnet precesses due to the Néel order parameter, it should be possible to detect the precession of the FM moment by standard magnetic resonance experiments. The addition of a second AF junction, as suggested above, should affect the ferromagnetic resonance.

V. GENERALIZED JOSEPHSON EFFECTS BETWEEN SYSTEMS WHERE ANTIFERROMAGNETISM AND *d*-WAVE SUPERCONDUCTIVITY COEXIST

In this section, we apply the same approach to study tunneling currents across a junction between two materials that are d-wave superconducting antiferromagnets, i.e., with homogeneous coexistence of antiferromagnetism and d-wave superconductivity. Since in the end we expect symmetry considerations instead of details of the Hamiltonian to be the dominant effect, we start with the simplest mean-field model that yields coexistence of the two order parameters. Then we show that for coexisting orders, the ordinary Josephson effect can be modulated by the antiferromagnetism and that the spin-Josephson effect can be modulated by the superconductivity. These effects occur already to leading order in the square of the tunneling matrix element.

A. Model

To take the simplest possible case that has the appropriate symmetries, we model the AF/SC coexistence on each side of the junction by the following phenomenological mean-field Hamiltonian:[20]

$$\hat{H}=\sum_{\mathbf{k}\sigma}(\xi_{\mathbf{k}}-\mu)c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+U\sum_i(c^{\dagger}_{i,\uparrow}c_{i,\uparrow}\langle c^{\dagger}_{i,\downarrow}c_{i,\downarrow}\rangle+\text{H.c.})-V\sum_i(\Delta^{\dagger}_{d,i}\langle\Delta_{d,i}\rangle+\text{H.c.})-W\sum_i(\Delta^{\dagger}_{t,i}\langle\Delta_{t,i}\rangle+\text{H.c.}). \tag{34}$$

The possibility of a spin-triplet pair amplitude $\langle\Delta_{t,i}\rangle$ must also be considered in general for self-consistency.[21] The singlet $\Delta_{d,i}$ and triplet $\Delta_{t,i}$ pair destruction operators are defined as

$$\Delta_{d,i}=\frac{1}{2}\sum_{\delta}g(\delta)(c_{i+\delta,\uparrow}c_{i,\downarrow}-c_{i+\delta,\downarrow}c_{i,\uparrow}), \tag{35}$$

$$\Delta_{t,i}=\frac{1}{2}\sum_{\delta}g(\delta)(c_{i+\delta,\uparrow}c_{i,\downarrow}+c_{i+\delta,\downarrow}c_{i,\uparrow}) \tag{36}$$

and the structure factor $g(\delta)$ is chosen to have a d-wavelike form such as

$$g(\delta)=\begin{cases}1/2 & \text{if}\ \ \delta=(\pm1,0)\\ -1/2 & \text{if}\ \ \delta=(0,\pm1)\\ 0 & \text{if}\ \ \text{otherwise}.\end{cases} \tag{37}$$

The strength of AF, SC, and spin-triplet pair interactions is governed by U, V, and W, respectively, which are all positive in our study. $\xi_{\mathbf{k}}$ is a tight-binding energy dispersion given as $\xi_{\mathbf{k}}=-2t(\cos k_x+\cos k_y)-4t'\cos k_x\cos k_y$, where t and t' are hopping constants for nearest neighbors and next-nearest neighbors, and μ is the chemical potential, which controls the particle density n.

For completeness, we recall the results of Ref. 20 that we need. In the mean-field approximation, three different order parameters m, s, and t corresponding to the three different interactions may be defined in the following way:

$$\langle c^{\dagger}_{i\sigma}c_{i\sigma}\rangle=\langle n_{i\sigma}\rangle=\frac{n}{2}+\sigma m\cos(\mathbf{Q}\cdot\mathbf{r}_i), \tag{38}$$

$$\frac{1}{2}\sum_{\delta}g(\delta)\langle c_{i+\delta,\uparrow}c_{i,\downarrow}-c_{i+\delta,\downarrow}c_{i,\uparrow}\rangle=s, \tag{39}$$

$$\frac{1}{2}\sum_{\delta}g(\delta)\langle c_{i+\delta,\uparrow}c_{i,\downarrow}+c_{i+\delta,\downarrow}c_{i,\uparrow}\rangle=t\cos(\mathbf{Q}\cdot\mathbf{r}_i), \tag{40}$$

where $\mathbf{Q}$ is the (commensurate) antiferromagnetic wave vector equal to (π,π) in two dimensions. Now the mean-field Hamiltonian H_{MF} is quadratic in the original electron operator and in terms of a new four-component field operator $\psi_{\mathbf{k}}$ it becomes bilinear

$$\hat{H}_{\rm CM} = \sum_{\mathbf{k}}^{*} \psi_{\mathbf{k}}^{\dagger} M_{\mathbf{k}} \psi_{\mathbf{k}} + E_0, \tag{41}$$

where

$$\psi_{\mathbf{k}}^{\dagger} \equiv (c_{\mathbf{k}\uparrow}^{\dagger}, c_{-\mathbf{k}\downarrow}, c_{\mathbf{k}+\mathbf{Q}\uparrow}^{\dagger}, c_{-\mathbf{k}-\mathbf{Q}\downarrow}). \tag{42}$$

The matrix $M_{\mathbf{k}}$ is given as

$$M_{\mathbf{k}} = \begin{pmatrix} \epsilon_{\mathbf{k}} & Vs\phi(\mathbf{k}) & -Um & Wt\phi(\mathbf{k}) \\ Vs\phi(\mathbf{k}) & -\epsilon_{\mathbf{k}} & -Wt\phi(\mathbf{k}) & -Um \\ -Um & -Wt\phi(\mathbf{k}) & \epsilon_{\mathbf{k}+\mathbf{Q}} & -Vs\phi(\mathbf{k}) \\ Wt\phi(\mathbf{k}) & -Um & -Vs\phi(\mathbf{k}) & -\epsilon_{\mathbf{k}+\mathbf{Q}} \end{pmatrix}, \tag{43}$$

where

$$\epsilon_{\mathbf{k}} = \xi_{\mathbf{k}} - \mu \tag{44}$$

and $\phi(\mathbf{k}) = \cos k_x - \cos k_y$ is the Fourier transform of $g(\delta)$. The constant energy shift E_0 depending on m, s, t is given as

$$E_0 = N(Um^2 + Vs^2 + Wt^2 - \mu), \tag{45}$$

where N is the total number of lattice sites. The energy eigenvalues of $M_{\mathbf{k}}$ yield four energy dispersions $\pm E_{\pm}(\mathbf{k})$ [with $E_{\pm}(\mathbf{k}) > 0$] (Refs. 20 and 22),

$$E_{\pm}(\mathbf{k}) = \{(\epsilon_{\mathbf{k}}^2 + \epsilon_{\mathbf{k}+\mathbf{Q}}^2)/2 + (Um)^2 + [Vs\phi(\mathbf{k})]^2 + [Wt\phi(\mathbf{k})]^2 \pm g(\mathbf{k})\}^{1/2}, \tag{46}$$

where $g(\mathbf{k})$ is given as

$$g(\mathbf{k}) = \{(\epsilon_{\mathbf{k}}^2 - \epsilon_{\mathbf{k}+\mathbf{Q}}^2)^2/4 + (\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{Q}})^2[Wt\phi(\mathbf{k})]^2 + \{(\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{k}+\mathbf{Q}})(Um) + 2[Vs\phi(\mathbf{k})][Wt\phi(\mathbf{k})]\}^2\}^{1/2}. \tag{47}$$

When $s=t=0$ or $m=t=0$ the energy eigenvalues reduce to those obtained from the SDW or BCS factorizations of the corresponding interactions.

The transformation matrix A relating the original c operator to the eigenoperator γ is defined by

$$\psi_{i\mathbf{k}} = \sum_j A_{ij}(\mathbf{k}) \gamma_{j\mathbf{k}}. \tag{48}$$

All the elements of the matrix A are given in the appendix of Ref. 20. The columns of the matrix A are the eigenvectors of the mean-field Hamiltonian with the corresponding eigenenergies ordered as follows: $E_{\mathbf{k}}^1 = +E_+(\mathbf{k})$, $E_{\mathbf{k}}^2 = -E_+(\mathbf{k})$, $E_{\mathbf{k}}^3 = -E_-(\mathbf{k})$, $E_{\mathbf{k}}^4 = +E_-(\mathbf{k})$. The vacuum state of $\psi_{\mathbf{k}}$ is obtained by filling the vacuum of the original operators $c_{\mathbf{k}\sigma}$ with down spin electrons. As usual in the grand canonical ensemble, the ground state is then obtained by filling the vacuum of $\psi_{\mathbf{k}}$ with all the negative energy quasiparticles, i.e., with $\gamma_{2\mathbf{k}}^{\dagger}$ and $\gamma_{3\mathbf{k}}^{\dagger}$ whose energies, $E_{\mathbf{k}}^2 = -E_+(\mathbf{k})$ and $E_{\mathbf{k}}^3 = -E_-(\mathbf{k})$, respectively, are negative for all $\mathbf{k}$. One recovers the BCS and SDW ground states in the appropriate limits. In general that ground state contains triplet pairs.

B. Combined spin-Josephson effect and ordinary Josephson effect in superconducting antiferromagnets

We now turn to the calculation of the spin and charge tunneling current across the junction. In terms of the γ's, the equation for $d\hat{\mathbf{S}}_L/dt$, Eq. (11), can be rewritten as

$$\begin{aligned}\dot{\mathbf{S}}_G(t) = \frac{1}{N}\sum_{\mathbf{k},\mathbf{q}}^{*}\sum_{\beta}\sum_{i,j} \mathrm{Im}[\exp i\Delta\phi/2 t_{\mathbf{k}\mathbf{q}} &\times\{\vec{\sigma}_{\uparrow\beta}\mathbf{U}_{\beta\uparrow}\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\uparrow\uparrow}\langle\gamma^{\dagger}_{i\mathbf{k}}(t)\gamma_{j\mathbf{q}}(t)\rangle \\ &+\vec{\sigma}_{\uparrow\beta}\mathbf{U}_{\beta\downarrow}\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\uparrow\downarrow}\langle\gamma^{\dagger}_{i\mathbf{k}}(t)\gamma^{\dagger}_{j\mathbf{q}}(t)\rangle \\ &+\vec{\sigma}_{\downarrow\beta}\mathbf{U}_{\beta\uparrow}\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\downarrow\uparrow}\langle\gamma_{i\mathbf{k}}(t)\gamma_{j\mathbf{q}}(t)\rangle \\ &+\vec{\sigma}_{\downarrow\beta}\mathbf{U}_{\beta\downarrow}\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\downarrow\downarrow}\langle\gamma_{i\mathbf{k}}(t)\gamma^{\dagger}_{j\mathbf{q}}(t)\rangle\}],\end{aligned} \tag{49}$$

where the phase difference and the difference in quantization axis have been gauged into the tunneling matrix element and where we defined

$$\begin{aligned}\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\uparrow\uparrow} &= [A_{1i}(\mathbf{k}) + A_{3i}(\mathbf{k})][A_{1j}(\mathbf{q}) + A_{3j}(\mathbf{q})], \\ \tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\uparrow\downarrow} &= [A_{1i}(\mathbf{k}) + A_{3i}(\mathbf{k})][A_{2j}(\mathbf{q}) + A_{4j}(\mathbf{q})], \\ \tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\downarrow\uparrow} &= [A_{2i}(\mathbf{k}) + A_{4i}(\mathbf{k})][A_{1j}(\mathbf{q}) + A_{3j}(\mathbf{q})], \\ \tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\downarrow\downarrow} &= [A_{2i}(\mathbf{k}) + A_{4i}(\mathbf{k})][A_{2j}(\mathbf{q}) + A_{4j}(\mathbf{q})].\end{aligned} \tag{50}$$

We compute each average in Eq. (49) to first order in the tunneling Hamiltonian as in Eq. (12) which requires H_T to be also rewritten in terms of the γ's. However, unlike the previous case of itinerant antiferromagnetism, all the terms in H_T must now be included since unconventional pairing is allowed by the presence of superconductivity. There will appear two-point correlation functions of the form $\langle c^{\dagger}_{\mathbf{k}\sigma}(t)c^{\dagger}_{-\mathbf{k}-\sigma}(t')d_{\mathbf{q}\sigma}(t)d_{-\mathbf{q}-\sigma}(t')\rangle_0$, corresponding to the coherent tunneling of Cooper pairs, and these will acquire a factor $\exp(i\Delta\varphi)$, where $\Delta\varphi$ is the phase difference between the superconducting order parameters of the two leads. Taking this into account, the calculation of the staggered magnetic moment current leads to

$$\dot{\mathbf{S}}_G(t) = (I_c + J_c \cos\Delta\varphi)\hat{s}_D \times \hat{s}_G, \tag{51}$$

where

$$\begin{aligned}I_c \equiv \frac{1}{N^2}\sum_{\substack{\mathbf{k},\mathbf{q}\\ i,j}}^{*} |t_{\mathbf{k}q}|^2 \Bigg\{ &[(\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\uparrow\uparrow})^2 + (\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\downarrow\downarrow})^2] P\left[\frac{f(E^j_{\mathbf{q}}) - f(E^i_{\mathbf{k}})}{E^j_{\mathbf{q}} - E^i_{\mathbf{k}}}\right] \\ &+ [(\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\uparrow\downarrow})^2 + (\tilde{\Gamma}^{ij}_{\mathbf{k}\mathbf{q}\downarrow\uparrow})^2] P\left[\frac{1 - f(E^j_{\mathbf{q}}) - f(E^i_{\mathbf{k}})}{E^j_{\mathbf{q}} + E^i_{\mathbf{k}}}\right]\Bigg\}\end{aligned} \tag{52}$$

and

$$J_c \equiv \frac{1}{N^2}\sum_{\substack{\mathbf{k},\mathbf{q}\\ i,j}}^{*} |t_{\mathbf{kq}}|^2 \tilde{\Gamma}^{ij}_{\mathbf{kq}\uparrow\uparrow}\tilde{\Gamma}^{ij}_{\mathbf{kq}\downarrow\downarrow} P\left[\frac{f(E^j_{\mathbf{q}})-f(E^i_{\mathbf{k}})}{E^j_{\mathbf{q}}-E^i_{\mathbf{k}}} + \frac{1-f(E^j_{\mathbf{q}})-f(E^i_{\mathbf{k}})}{E^j_{\mathbf{q}}+E^i_{\mathbf{k}}}\right] \qquad (53)$$

with $E^i_{\mathbf{k}}$ defined at the end of the previous subsection.

Similarly, we obtain the charge supercurrent (or Josephson current) across the junction by computing the statistical average of the time derivative of the number operator on the left lead, $N_G = \Sigma_{\mathbf{k}\sigma} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$,

$$\dot{N}_G(t) = (\bar{I}_c + \bar{J}_c \cos\theta)\sin\Delta\varphi, \qquad (54)$$

with

$$\bar{I}_c \equiv \frac{2}{N^2\hbar}\sum_{\substack{\mathbf{k},\mathbf{q}\\ i,j}}^{*} |t_{\mathbf{kq}}|^2 \tilde{\Gamma}^{ij}_{\mathbf{kq}\uparrow\uparrow}\tilde{\Gamma}^{ij}_{\mathbf{kq}\downarrow\downarrow} P \times\left[\frac{f(E^j_{\mathbf{q}})-f(E^i_{\mathbf{k}})}{E^j_{\mathbf{q}}-E^i_{\mathbf{k}}} - \frac{1-f(E^j_{\mathbf{q}})-f(E^i_{\mathbf{k}})}{E^j_{\mathbf{q}}+E^i_{\mathbf{k}}}\right], \qquad (55)$$

and $\bar{J}_c$ is obtained from $\bar{I}_c$ by changing the sign between the two quotients in the brackets. Note that $\bar{J}_c = 2J_c/\hbar$. We discuss the consequences of this equality in the next section.

VI. DISCUSSION AND CONCLUSION

The robustness of the spin-Josephson effect is related to the rigidity of a broken symmetry, in analogy with the ordinary Josephson effect. From a phenomenological point of view, the common thread between the generalized Josephson effects described in this work is that the effective Ginzburg-Landau free energy, or the effective action, contains interaction terms proportional to product of the order parameters on the left and on the right of the tunnel junction. Symmetry imposes $(\Psi_L\Psi_R^* + \text{H.c.}) \propto \cos(\Delta\phi)$ in the superconducting case and $\mathbf{S}_L(\mathbf{Q})\cdot\mathbf{S}_R(-\mathbf{Q})$ in the antiferromagnetic case, with $\mathbf{S}_L(\mathbf{Q})$ the Néel vector. The latter case is completely analogous to the ferromagnetic one where one finds the scalar product of the uniform magnetizations.[4–6] The corresponding equations of motion involve $\sin(\Delta\phi)$ in the superconducting case and its natural generalization in the vector case, namely, the cross product. The resulting antiferromagnetic spin current leads to a precession of the order parameters about their sum on either side of the junction. In the case of a spin 1/2, this precession can be detected experimentally by comparing the magnetic resonance of the ferromagnet in an AF/I/FM junction with the magnetic resonance of a AF/I/AF/I/FM junction, as we have shown. We speculate that similar effects exist for spin one antiferromagnetic junctions in which case the effect could be directly observed through antiferromagnetic resonance.[19]

In the mean-field solution, the direction of the triplet in spin space is locked to that of the antiferromagnetic order parameter. For a d-wave superconducting antiferromagnet, there is an additional cross term in the Ginzburg-Landau free energy that comes from the triplet component $\vec{\Psi}^t$ of the order parameter, namely, $(\vec{\Psi}^t_L\cdot\vec{\Psi}^{t*}_R + \text{H.c.})$ that leads to (a) a modulation of the critical charge current by the dot product of the antiferromagnetic order parameters and (b) a modulation of the critical spin current by the superconducting phase difference. An analogous effect was found for superconducting ferromagnets.[8] The equality $\bar{J}_c = 2J_c/\hbar$ found at the end of the previous section is a direct consequence of the fact that both cross terms come from a single term in the effective Ginzburg-Landau free energy. There is thus a reciprocity between the modulation of the Josephson charge current by the antiferromagnetic order parameter and the modulation of the antiferromagnetic Josephson spin current by the superconducting order parameter.

An additional analogy between the ordinary Josephson effect and the spin-Josephson effect resides in gauge transformations. In the ordinary Josephson effect, a gauge transformation allows the phase difference to be reported entirely on the tunneling matrix element. Similarly, in the spin-Josephson effect, one can make a choice of quantization axis (analogous to a gauge choice) that transfers to the tunneling matrix element everything that is related to the difference in order-parameter orientation across the junction.

We learned from the microscopic calculation that tunneling of Cooper pairs is replaced by tunneling of the corresponding condensed objects in the antiferromagnetic case, namely, spin-one, charge-zero, particle-hole pairs, with zero projection along the direction of the antiferromagnetic order parameter. Alternatively, and also in analogy with the superconducting case, the terms that lead to the spin-Josephson effect represent interference in the tunneling process between momentum $\mathbf{k}+\mathbf{Q}$ spin-up particles and momentum $-\mathbf{k}$ spin-down hole. In superconducting antiferromagnets, both types of condensed objects, Cooper pairs and finite $\mathbf{Q}$ particle-hole pairs, are present hence one sees both Josephson and spin-Josephson effects. In addition, in superconducting antiferromagnets a pure singlet state between nearest neighbors is impossible because of the broken spin symmetry between the two sublattices. Hence there are always triplet Cooper pairs with total wave vector $\mathbf{Q}$.[21] It is the tunneling of this type of Cooper pair that leads to the cross term between charge- and spin-Josephson effects mentioned above. Indeed, a triplet Cooper pair carries both phase and spin information and hence its tunneling can influence both spin and charge and can be manipulated by fields that couple to either spin or charge.

One of the most far reaching consequences of the Josephson effect in ordinary superconductors is the ac effect where phase difference between the superconductors increases linearly with time due to an applied voltage difference. The corresponding frequency of the current is proportional to the voltage drop through a material-independent universal constant $2e/h$. In the case of the spin-Josephson effect, even without external field there is a precession frequency. In the presence of a uniform magnetic field, the two Néel vectors precess around their sum while that sum precesses around the magnetic field. The correct analog of the ac effect corresponds to adding fields that will lead to a time variation in the angle between the two order parameters, the analog of the phase difference. Adding a uniform magnetic field does not achieve this. It is only in the presence of a magnetic field difference between the two sides of the junction that the angle becomes time dependent, as described by Eq. (29), $\dot{\vartheta}(t) = -g\mu_B \delta\mathbf{B}\cdot\hat{\mathbf{e}}(t)$ with $\delta\mathbf{B}$ the magnetic field difference and $\hat{\mathbf{e}}(t) \equiv \hat{\mathbf{s}}_R(t)\times\hat{\mathbf{s}}_L(t)$. If the correct order parameters are used, this result is valid for tunneling between ferromagnets as well. As far as we know, no analog if the equation $\dot{\vartheta}(t) = -g\mu_B \delta\mathbf{B}\cdot\hat{\mathbf{e}}(t)$ has appeared in the literature on spin-Josephson effect. Contrary to the ordinary Josephson effect, the proportionality constant in front of $\delta\mathbf{B}$ is not universal since g is material dependent. This is because the coupling of the magnetic field to matter is not a pure gauge coupling. In the presence of spin-orbit interactions, both charge and spin currents couple to the field.

One possible application of our work is to use it to differentiate between homogeneous coexistence of antiferromagnetism and d-wave superconductivity and phase separation between these two types of order. It would probably be easiest to measure time-dependent modulation of the "charge" Josephson critical current induced by a junction where a magnetic field gradient small enough not to destroy superconductivity is applied to the tunnel junction. In the case of phase separation, the cross terms that we described are not present since they come from tunneling of a finite momentum triplet Cooper pair that exists only when there is homogeneous coexistence. The issue of homogeneous coexistence is important for the cuprates, the heavy fermions, the organics and the pnictides where antiferromagnetism and d-wave superconductivity often seem to overlap in the phase diagram.

Further possible generalizations of our work include (a) studying tunneling between states with more complicated order parameters. (b) Calculating the analogs of the effects that we found in the case where the tunnel junction is replaced by a normal metal. It is known[23–25] that Andreev states in the superconducting case profoundly change the dependence of the current on the phase difference but not its periodicity. Since the analog of Andreev states exist in the antiferromagnet,[14] similar effects are expected for the spin-Josephson effect.

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